

TENSOR PRODUCTS FOR THE DESITTER GROUP

BY

ROBERT P. MARTIN

ABSTRACT. The decomposition of the tensor product of a principal series representation with any other irreducible unitary representation of G is determined for the simply connected double covering, $G = \text{Spin}(4, 1)$, of the DeSitter group.

Introduction. Let $G = \text{Spin}(4, 1)$ be the simply connected double covering of the DeSitter group, $G = KAN$ an Iwasawa decomposition of G , M the centralizer of A in K , and $P = MAN$ the associated minimal parabolic subgroup. The finite-dimensional irreducible representations S of P (nonunitary as well as unitary) and the analysis of the induced representations $\text{Ind}_P^G S$, via the intertwining operators of Knapp and Stein in [8] and Kunze and Stein in [9], have played key roles in determining the unitary dual, $\hat{G}$, of G . In this paper, we show how knowledge of the decompositions of the induced representations $\text{Ind}_P^G T$ for the infinite-dimensional irreducible unitary representations T of P enables us to determine the decomposition of the tensor product $\pi(n, s) \otimes \pi$ into irreducibles where $\pi(n, s)$ is a principal series representation of G and $\pi \in \hat{G}$ is arbitrary. Qualitatively, we expect similar results for semisimple Lie groups in general.

The main results concerning the decomposition of tensor products appear in §4. In each case, $\pi(n, s) \otimes \pi$ decomposes into a direct sum of the form $T_c \oplus T_d$, where T_c is a continuous direct sum with respect to Plancherel measure on $\hat{G}$ of representations from the principal series of G , and T_d is a discrete direct sum of representations from the discrete series of G . The multiplicities of principal and discrete series representations appearing in this decomposition are all finite and depend only upon n and the restriction of π to P . The range of the multiplicity function for this decomposition is finite.

Briefly, the problem of decomposing $\pi(n, s) \otimes \pi$ “reduces” to three subproblems: (1) that of decomposing the restriction to P of each $\pi \in \hat{G}$, $(\pi)_P$; (2) that of decomposing the tensor product of a finite-dimensional irreducible unitary representation of P with an infinite-dimensional irreducible unitary representation of P ; and (3), that of decomposing $\text{Ind}_P^G T$ for each infinite-dimensional irreducible unitary representation T in $\hat{P}$ (the unitary dual of P). Subproblem two follows easily by using Mackey’s tensor product theorem. In light of our knowledge of Plancherel measures on $\hat{P}$ and $\hat{G}$ and the Mackey-Anh reciprocity theorem (see [1]), subproblem three is essentially equivalent to subproblem one. The main obstacle for determining the decomposition of $\pi(n, s) \otimes \pi$ is subproblem one. If π is a principal series representation of G , the decomposition of $(\pi)_P$ is easily found by using Mackey’s subgroup

Received by the editors November 19, 1982 and, in revised form, August 18, 1983.

1980 *Mathematics Subject Classification*. Primary 22E43, 81C40.

©1984 American Mathematical Society
 0002-9947/84 \$1.00 + \$.25 per page

theorem and is well known. The problem of decomposing $(\pi)_P$ for nonprincipal series representations in $\hat{G}$ is more difficult and was determined by R. Fabec in [6] by computing the Fourier transforms of the Kunze-Stein intertwining operators in the noncompact realization for the nonunitary principal series of G . In each case, $(\pi)_P$ decomposes into a finite direct sum of infinite-dimensional representations in $\hat{P}$, each occurring with multiplicity one. Recent results by A. Breda in [3] on Fourier transforms of intertwining operators are quite encouraging and suggest that these results will be fruitful in solving subproblem one (and hence for decomposing tensor products) for semisimple Lie groups in general.

The decomposition of the tensor product of two principal series representations for the DeSitter group was determined in [13] (the case of two class one principal series representations also appears in [5]). The basic methodology of the approach used in [13] originates in the paper of G. Mackey [10] for the group $SL(2, \mathbb{C})$ and continues in the papers of N. Anh [1] for $SL(n, \mathbb{C})$, F. Williams [21] for complex semisimple Lie groups, and R. Martin [11] for real-rank one semisimple Lie groups. These papers decompose the tensor product of two principal series representations, for the groups G in question, by determining the restriction of representations in $\hat{G}$ to the subgroup MA . The idea of focusing ones attention on restricting representations in $\hat{G}$ to MAN , and hence being able to decompose $\pi(n, s) \otimes \pi$, where π is not necessarily a principal series representation, originates in [12] for the groups $SL(2, R)$, $SL(2, \mathbb{C})$ and $SL(2, k)$, where k is a locally compact, nondiscrete, totally disconnected field whose residual characteristic is not two. The paper of L. Pukánszky [14] determines the decomposition of the tensor product of two principal or complementary series representations for the group $SL(2, R)$. A complete list of all the possible decompositions of tensor products of representations in $\hat{G}$ when $G = SL(2, R)$ appears in the paper of J. Repka [15].

1. The structure of G . In this section we summarize the main results concerning the structure of the DeSitter group and its two-fold covering $Spin(4, 1)$ that we shall use in this paper. Further details may be found in [5, 6 or 17].

Let $O(4, 1)$ denote the group of linear transformations of R^5 which preserve the quadratic form $-x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2$. If J is the diagonal matrix $J = [-1, 1, 1, 1, 1]$, then $O(4, 1)$ may be identified with $\{g \in GL(5, R): {}^t g J g = J\}$. The connected component of the identity is the group

$$G' = SO_e(4, 1) = \{g \in O(4, 1): \det(g) = 1, g_{00} \geq 1\}$$

which is commonly referred to as the DeSitter group. G' is a connected semisimple real-rank one Lie group with trivial center. We let $G = Spin(4, 1)$ denote the simply-connected double covering of G' . As indicated in [17], we may realize G as a certain collection of two-by-two matrices over the quaternions. If F denotes the division ring of quaternions $x = x_1 + x_2 i + x_3 j + x_4 k$, where x_1, x_2, x_3, x_4 are reals, $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, $jk = -kj = i$, $ki = -ik = j$, and we let $\bar{x} = x_1 - x_2 i - x_3 j - x_4 k$, and $|x| = \sqrt{x\bar{x}}$, then G is isomorphic to the group

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in F, \bar{a}b = \bar{c}d, |a|^2 - |c|^2 = 1, |d|^2 - |b|^2 = 1 \right\}.$$

The group $U = \{x \in F: |x| = 1\}$ of unit quaternions is easily seen to be isomorphic to $SU(2)$ [via the mapping $x_1 + x_2 i + x_3 j + x_4 k \mapsto \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$, $a = x_1 + x_2 i$, $b = x_3 + x_4 i$]

for example] and if we let

$$K = \left\{ \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} : u, v \in U \right\} \approx \mathrm{SU}(2) \times \mathrm{SU}(2) \approx \mathrm{Spin}(4),$$

$$A = \left\{ \begin{pmatrix} \mathrm{ch}(1/2t) & \mathrm{sh}(1/2t) \\ \mathrm{sh}(1/2t) & \mathrm{ch}(1/2t) \end{pmatrix} = a_t : t \in R \right\} \approx R^+,$$

and

$$N = \left\{ \begin{pmatrix} 1-x & x \\ -x & 1+x \end{pmatrix} = n_x : x = 1/2(x_2i + x_3j + x_4k) \right\} \approx R^3,$$

then $G = KAN$ is an Iwasawa decomposition of G . If M denotes the centralizer of A in K , then $M = \left\{ \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix} = m_u : u \in U \right\} \approx \mathrm{Spin}(3)$ and $P = MAN$ is a (minimal parabolic) subgroup of G which contains N as a normal subgroup. One easily computes that the actions of M and A on N are given by

$$m_u n_x m_u^{-1} = m_u \cdot n_x = n_{uxu}, \quad a_t n_x a_t^{-1} = a_t \cdot n_x = n_{(e^t)x},$$

i.e., M acts by rotations and A acts by dilations. If $\tilde{M}$ denotes the normalizer of A in K , then the Weyl group $W = \tilde{M}/M$ has order two and we may take $e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $w = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ as representatives of the cosets of W . In addition to the Iwasawa decomposition of G , one has the KAK decomposition (see [17, p. 366]) and the Bruhat decomposition $G = PeP \cup PwP$ (and so there are only two $P : P$ double cosets in G with only one of positive Haar measure). Setting

$$V = \left\{ \begin{pmatrix} 1-x & -x \\ x & 1+x \end{pmatrix} = v_x : x = 1/2(x_2i + x_3j + x_4k) \right\}$$

and using the relations $w^{-1}Aw = A$, $w^{-1}Mw = M$, the latter decomposition can be expressed as $G = Pw^{-1} \cup PV$ and so up to a manifold of lower dimension (and so a set of Haar measure zero) we see that $G = PV$.

If $H:G \rightarrow G'$ denotes the homomorphism on p. 366 of [17] and we let $K' = H(K)$, $A' = H(A)$, $N' = H(N)$, $M' = H(M)$ and $V' = H(V)$, then

$$\ker(H) = \text{the center of } G = Z(G) = \left\{ \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix} \right\} \subseteq K$$

and so we have the following isomorphisms: $K/Z(G) \approx K' \approx \mathrm{SO}(4)$, $M/Z(G) \approx M' \approx \mathrm{SO}(3)$, $A \approx A' \approx R^+$ and $N \approx N' \approx V \approx V' \approx R^3$. It is easily seen that

$$\begin{aligned} K' &= \left\{ \begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix} : k \in \mathrm{SO}(4) \right\}, \\ A' &= \left\{ \begin{pmatrix} \mathrm{ch} a & \mathrm{sh} a & 0 \\ \mathrm{sh} a & \mathrm{ch} a & 0 \\ 0 & 0 & I \end{pmatrix} : a \in R \right\}, \\ M' &= \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & u \end{pmatrix} : u \in \mathrm{SO}(3) \right\}. \end{aligned}$$

The Lie algebra $\mathcal{G}$ of G (and of G') is described in [4, 5, 6, or 17].

2. The representation theory of G . In this section we describe the representation theory of G and its subgroups. We begin by looking at the representation theories of M , K , A and $C = MA$. $M = \text{Spin}(3) \approx \text{SU}(2)$ and it is well known that $\hat{M} = \{\sigma^k: k = 0, 1/2, 1, \dots\}$ where each σ^k acts on a space V^k of dimension $2k + 1$ (see p. 110 of [19]). For $k = 0, 1, \dots$, σ^k is a single-valued representation of $M' \approx \text{SO}(3)$ while each σ^k , $k = 1/2, 3/2, \dots$, is a double-valued representation of M' . The group $T = \{m_u: u = e^{i\theta}, \theta \in R\}$ is a torus in M and we shall view $\hat{T} = \{\tau^n: n = 0, \pm 1/2, \pm 1, \dots\}$ where $\tau^n(m_u) = (e^{i\theta})^{2n}$. One sees easily, using the isomorphism $M \approx \text{SU}(2)$ and the realization of σ^k given on p. 110 of [19], that the restriction of σ^k to T decomposes as

$$(\sigma^k)_T \simeq \tau^{-k} \oplus \tau^{-k+1} \oplus \dots \oplus \tau^k \quad \text{for } k = 0, 1/2, 1, \dots$$

Since $K = \text{Spin}(3) \times \text{Spin}(3) \approx \text{Spin}(4)$, we have that $\hat{K} = \{\sigma^{k,k'}: k, k' = 0, 1/2, 1, \dots\}$ where each $\sigma^{k,k'}$ acts on $V^{k,k'} = V^k \times V^{k'}$. It is also known that the restriction of $\sigma^{k,k'}$ to M decomposes as

$$(\sigma^{k,k'})_M \simeq \bigoplus_{j=|k-k'|}^{k+k'} \sigma^j$$

[19, p. 175]. The representation $\sigma^{k,k'}$ is a single- or double-valued representation of K' according to whether $k + k'$ is integral or not.

The irreducible representations (quasicharacters) of A are given by $\lambda^s(a_t) = e^{ist}$ for s in $\mathbf{C}$. These are unitary iff s is real and so we shall view $\hat{A} = \{\lambda^s: s \in R\}$. Since the group $C = MA$ is a direct product, $\hat{C} = \hat{M} \times \hat{A}$ with Plancherel measure μ_C on $\hat{C}$ being the product of the Plancherel measures on $\hat{M}$ and $\hat{A}$.

The group P is the semidirect product of MA with the closed normal subgroup N and so we can use the Mackey theory (see [20]) to describe the representation theory of P . After identifying $\hat{N}$ with R^3 , the action of MA on $\hat{N}$ is via rotation and dilation on R^3 . Thus there are just two orbits in $\hat{N}$ under this action—the zero orbit $\{(0, 0, 0)\}$ and everything else, O . The stability group corresponding to the zero orbit is MA itself while the stability group corresponding to the point $(1, 0, 0)$ in O is the group $\left\{ \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix} : u = x_1 + x_2 i, |u| = 1 \right\}$, i.e., our torus T in M . So, corresponding to the zero orbit we get the finite-dimensional irreducible unitary representations of P lifted from those of MA , i.e., the representations $(\sigma^k \times \lambda^s)'(man) = \sigma^k(m)\lambda^s(a)$ for $k = 0, 1/2, 1, \dots$ and s in R . These are the representations which, after inducing, give rise to the principal series of G . If we denote by β_1 the representation in $\hat{N}$ corresponding to $(1, 0, 0)$ in the orbit O , we obtain, corresponding to the nonzero orbit O , the family of infinite-dimensional irreducible unitary representations $T^n = \text{Ind}_{TN}^P(\tau^n \times \beta_1)$ for $n = 0, \pm 1/2, \pm 1, \dots$. We shall refer to the representations T^n for $n = 0, \pm 1/2, \pm 1, \dots$ as the generic representations of P . These are the representations, as we shall see, which, after inducing, provide information about certain tensor products on G . Thus we may view $\hat{P}$ as being

$$\hat{P} = \{(\sigma^k \times \lambda^s)': k = 0, 1/2, \dots, s \in R\} \cup \{T^n: n = 0, \pm 1/2, \dots\}.$$

If μ_P denotes Plancherel measure on $\hat{P}$, then the results of [7] show that μ_P is an atomic measure whose support is the set of generic representations of P .

We now describe the irreducible unitary representations of G . We follow the notation of [6] (although we make some minor changes in order to have the notation

resemble that of [4] more closely). Further details may be found in [4, 5, 17 or 18].

If $\sigma^p \in \hat{M}$ and λ^s is a quasicharacter of A , we let $(\sigma^p \times \lambda^s)'$ denote the representation of P given by $(\sigma^p \times \lambda^s)'(man) = \sigma^p(m)\lambda^s(a)$ and set $\pi(p, s) = \text{Ind}_P^G(\sigma^p \times \lambda^s)'$ acting on the Hilbert space $H(p, s)$. Each $\pi \in \hat{G}$ occurs as a subrepresentation of some $\pi(p, s)$ and, roughly speaking, the process of finding $\hat{G}$ involves determining which of the representations $\pi(p, s)$ can be “unitarized”, determining which are irreducible, decomposing the reducible ones, and determining the equivalences. In each of these steps, the intertwining operators of [8 and 9] play a key role.

When s is real, the representation $\pi(p, s)$ is unitary and it is known that $\pi(p, s)$ is reducible iff $s = 0$ and $p = 1/2, 3/2, \dots$ (see [8]). For $p = 1/2, 3/2, \dots$, $\pi(p, 0)$ splits into the direct sum of two irreducibles which we shall denote by $\pi^\pm(p, 1/2)$. It is well known that $\pi(p, s)$ is equivalent to $\pi(p, -s)$ and that $\pi(p, s)$ is not equivalent to $\pi(q, t)$ if $p \neq q$ or $|s| \neq |t|$. The collection $\hat{G}_p = \{\pi(p, s): s \geq 0, p = 0, 1/2, \dots\}$ is called the principal series of G . We will write $\hat{G}_i = \{\pi(p, s) \in \hat{G}_p: s \neq 0 \text{ if } p = 1/2, 3/2, \dots\}$ for the irreducible principal series and $\hat{G}_r = \{\pi^\pm(p, 1/2): p = 1/2, 3/2, \dots\}$ for the collection of irreducibles arising from the reducible principal series representations. The representation $\pi(p, s) \in \hat{G}_p$ in our notation corresponds to the representation $W(p, s)$ in Fabec’s classification and to $\nu_{p, \sigma}$ in Dixmier’s classification with $\sigma = (1 + s^2)/4$. The representations $\pi^\pm(p, 1/2)$ correspond to the representations $w^\pm(p, 0)$ in Fabec’s notation and to either $\pi_{p/1, 2}^\pm$ or $\pi_{p, 1/2}^\mp$ in Dixmier’s notation (the exact correspondence is not known).

When s is a nonzero real, the representation $\pi(p, -is)$ is not unitary. However, the results of [6] (see also [8]) show that when $p = 0, 1, 2, \dots$, it is possible to define a new inner product on the Hilbert space $H(p, -is)$ for certain real values of s in a “critical interval” $0 < s < c_p$ (for $p = 0$, $c_p = 3$ while for $p = 1, 2, \dots$, $c_p = 1$) for which the action of $\pi(p, -is)$ is unitary. The extension of $\pi(p, -is)$ to the completion of $H(p, -is)$ with respect to this new inner product is an irreducible unitary representation of G which we also denote by $\pi(p, -is)$. The collection of unitary representations $\hat{G}_c = \{\pi(p, -is): 0 < s < 3 \text{ if } p = 0, 0 < s < 1 \text{ if } p = 1, 2, \dots\}$ obtained in this fashion are all pairwise inequivalent and constitute the representations in the complementary series of G . They correspond to the representations $W(p, -is)$ in the notation of [6] and to the representations $\nu_{p, \sigma}$ in the notation of [4] with $\sigma = (1 - s^2)/4$.

Representations of the form $\pi(0, s)$ for s in R or $\pi(0, -is)$ for $0 < s < 3$ are called class one representations since they contain the trivial representation when restricted to K (see [4]).

When $p = 0$ and $s_m = 2m + 1$ for $m = 1, 2, 3, \dots$, the results of [6] show that $H(0, -is_m)$ contains a unique invariant subspace under the action of $\pi(0, -is_m)$ and that it is possible to define a new inner product of this subspace for which the action of $\pi(0, -is_m)$ is unitary. The representation $\pi(0, -is_m)$ acting on the completion of this subspace with respect to this new inner product is an irreducible unitary representation which we denote by $\pi_{m, 0}$, using Dixmier’s notation. The collection $\hat{G}_e = \{\pi_{m, 0}: m = 1, 2, \dots\}$ obtained in this fashion are pairwise inequivalent and constitute what we shall refer to as the class of endpoint representations of G . The name and choice of notation for this collection of representations are motivated by the description of the topology on $\hat{G}$ which we shall give at the end of this section.

The endpoint representation $\pi_{m,0}$ corresponds to the representation $V(m/2)$ in the notation of [6].

When $p = 1, 3/2, 2, \dots$ and $s_q = 2q - 1$ for 1 or $3/2 \leq q \leq p$, the results of [6] show that $H(p, -is_q)$ contains two subspaces which are invariant under the action of $\pi(p, -is_q)$ and that it is possible to define a new inner product on these subspaces for which the action of $\pi(p, -is_q)$ is unitary. We denote the two unitary representations obtained by extending the action of $\pi(p, -is_q)$ to the completion of each of these subspaces with respect to this new inner product by $\pi^\pm(p, q)$. The collection of unitary representations $\hat{G}_d = \{\pi^\pm(p, q) : p = 1, 3/2, \dots, 1 \text{ or } 3/2 \leq q \leq p\}$ obtained in this fashion are all irreducible and pairwise inequivalent and is referred to as the discrete series of G . The representations in $\hat{G}_d$ are the only irreducible unitary representations of G which are square integrable (in fact, they are integrable for $q \geq 5/2$). The representation $\pi^\pm(p, q)$ corresponds to $V^\pm(p, p - q)$ in the notation of [6] and to either $\pi_{p,q}^\pm$ or $\pi_{p,q}^\mp$ (the exact correspondence is not known) in the notation of [4].

The above representations are single- or double-valued representations of G' according to whether $p = 0, 1, 2, \dots$ or not.

Thus, we may write $\hat{G} = \hat{G}_i \cup \hat{G}_r \cup \hat{G}_c \cup \hat{G}_d \cup \hat{G}_e \cup \{I\}$ where I is the trivial representation of G . Plancherel measure μ_G on $\hat{G}$ is supported in $\hat{G}_i \cup \hat{G}_r \cup \hat{G}_d$ (see [20, Vol. II]). On $\hat{G}_i \cup \hat{G}_r$ it is a continuous measure while $\mu_G(\pi) > 0$ for each π in $\hat{G}_d$.

The topology on $\hat{G}$ was determined in [2]. To describe this topology, it is helpful to have a parametrization of $\hat{G}$ as a subset of R^2 . We do this by using Dixmier's parametrization of $\hat{G}$. With this in mind, we identify each representation $\nu_{p,s}$ in the continuous series of G (for $p = 0, s > -2$; $p = 1, 2, \dots, s > 0$; and $p = 1/2, 3/2, \dots, s > 1/4$) with the point (p, s) in R^2 . These points correspond to representations in the complementary series for $p = 0, -2 < s < 1/4$ and $p = 1, 2, \dots, 0 < s < 1/4$ and to representations in the irreducible principal series for $p = 0, 1, 2, \dots, s \geq 1/4$ and $p = 1/2, 3/2, \dots, s > 1/4$. We identify each pair of representations $\pi^\pm(p, q)$ in $\hat{G}_r \cup \hat{G}_d$ with a pair of points at $(p, -q)$. We identify the representation $\pi_{p,0}$ in $\hat{G}_e$ with the point $(p, 0)$ (so that these points occur as endpoints of the various intervals comprising the complementary series of G , for $p = 1, 2, \dots$). We identify the trivial representation, I , with the endpoint of the class one complementary series, $(0, -2)$.

As described in [2], the topology on $\hat{G}$, as identified with the above set, is the same as that it inherits as a subset of R^2 with the following exceptions: the closure of any subset of $S = \hat{G}_i \cup \hat{G}_c$ that would ordinarily contain the point $(0, -2)$ must also contain both $(0, -2)$ and $(1, 0)$; the closure of any subset of S that would ordinarily contain $(p, 1/4)$ for $p = 1/2, 3/2, \dots$ must contain the pair of points at $(p, -1/2)$; and the closure of any subset of S that would ordinarily contain $(p, 0)$ for $p = 1, 2, \dots$ must contain the pair of points at $(p, -1)$ in addition to the point $(p, 0)$.

It will also be useful, when describing the results of Theorems 2 and 4, to identify $\hat{G}_p \cup \hat{G}_d$ with the following fibre space in R^3 : we identify the principal series representation $\pi(p, s)$ in $\hat{G}_p$ with the point $(p, 0, s)$; we identify the discrete series representation $\pi^-(p, q)$ with the point $(p, -q, 0)$.

3. Restricting to P and inducing from P . In Theorem 1 of this section we describe the results of R. Fabec in [6] concerning the restrictions of the various irreducible representations in $\hat{G}$ to the minimal parabolic subgroup P . These results, combined with the Mackey-Anh reciprocity theorem and previously known results concerning principal series representations, allow us to describe, in Theorem 2, the decomposition of $\text{Ind}_P^G T$ for any T in $\hat{P}$. In §4, we will first describe, in Theorem 4, the tensor product $\pi(n, s) \otimes \pi$ of a principal series representation with π in $\hat{G}$ in terms of representations of the form $\text{Ind}_P^G T$ for T in $\hat{P}$. Theorems 2 and 4 will then enable us to describe the decomposition of $\pi(n, s) \otimes \pi$ in terms of principal and discrete series representations of G .

THEOREM 1 [R. FABEC]. *If $G = \text{Spin}(4, 1)$ and $\pi \in \hat{G}$, then the restriction of π to the minimal parabolic subgroup P decomposes into a discrete direct sum of a finite number of generic representations of P , each occurring with multiplicity one. There are four cases to consider:*

- A. *If $\pi = \pi(m, \cdot)$ is a principal or complementary series representation of G , then $(\pi)_P \simeq T^{-m} \oplus T^{-m+1} \oplus \cdots \oplus T^m$.*
- B. *If $\pi = \pi_{m,0}$ is an endpoint representation of G , then $(\pi)_P \simeq T^0$.*
- C. *If $\pi = \pi^+(m, q)$ is a discrete or reducible principal series representation of G from the positive side, then $(\pi)_P \simeq T^q \oplus T^{q+1} \oplus \cdots \oplus T^m$.*
- D. *If $\pi = \pi^-(m, q)$ is a discrete or reducible principal series representation of G from the negative side, then $(\pi)_P \simeq T^{-m} \oplus T^{-m+1} \oplus \cdots \oplus T^{-q}$.*

Note. 1. The restriction of a principal or complementary series representation $\pi(m, s)$ to P is independent of the parameter s .

2. For a fixed m , the restriction to P of a representation of the type $\pi^\pm(m, \cdot)$ is a subrepresentation of the restriction to P of any of the principal or complementary series representations $\pi(m, \cdot)$.

3. The results on restricting irreducibles in $\hat{G}$ to P are compatible with the description of the topology on $\hat{G}$ given in §2. Using Dixmier's notation, we see for $m = 1/2, 3/2, \dots$ that the restrictions to P of the continuous series representations $\nu_{m,s}$, namely, $T^{-m} \oplus \cdots \oplus T^{-1/2} \oplus T^{1/2} \oplus \cdots \oplus T^m$, "approach" the sum of the restrictions to P of the two representations $\pi^+(m, 1/2)$ and $\pi^-(m, 1/2)$, as s approaches 0. For $m = 1, 2, \dots$, we see that the restrictions to P of the continuous series representations $\nu_{m,s}$, viz., $T^{-m} \oplus \cdots \oplus T^{-1} \oplus T^0 \oplus T^1 \oplus \cdots \oplus T^m$, approach the sum of the restrictions to P of the three representations $\pi^+(m, 1)$, $\pi^-(m, 1)$ and $\pi_{m,0}$, as s approaches 0. The results of [4] show that a similar statement can be made concerning the restrictions of irreducibles in $\hat{G}$ to the subgroup K . One can think of this as "continuity of restriction", although, for the case of the minimal parabolic subgroup P , this is actually a stronger result since the closure of a generic representation in $\hat{P}$ will also contain finite-dimensional representations in $\hat{P}$.

4. The representations $\pi^\pm(m, m)$ for $m = 1/2, 1, \dots$ are also irreducible when restricted to P (it is not hard to give alternate proofs of the irreducibility of these representations when restricted to P either by using the infinitesimal methods in [4] or the global realizations of these representations given in [17]) and, together with T^0 , the restrictions of these representations to P completely exhaust the set of generic representations of P . It may be of interest to note that these results suggest yet another way to realize the discrete series of G . For example, starting

with the representations $T^m, \dots, T^q$ of P , there is a unique K -action on $S = T^m \oplus \dots \oplus T^q$ which is compatible with the $P = MAN$ -action on S and which makes S an irreducible unitary representation of G (which must be $\pi^+(m, q)$ in light of its restriction to P). Similarly, starting with the representations $T^{-m}, \dots, T^{-q}$ of P , one should be able to obtain $\pi^-(m, q)$. If, in addition, one did this for complementary series and endpoint representations, one would be able to obtain $\hat{G}$ directly from $\hat{P}$.

5. One can find further results on restricting irreducibles to minimal parabolic subgroups for $\mathrm{SL}(2, R)$, $\mathrm{SL}(2, \mathbb{C})$ and $\mathrm{SL}(2, k)$ in [12]. The paper [16] contains results on restricting holomorphic discrete series representations for semisimple Lie groups in general to a minimal parabolic. In all of the above cases, the restrictions to P of the irreducibles in question decompose into a discrete direct sum of a finite number of generic representations of P with multiplicity one.

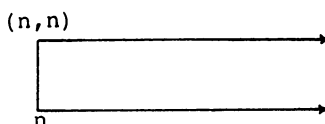
THEOREM 2. (i) If $(\sigma^n \times \lambda^s)'$, $n = 0, 1/2, 1, \dots, s \in R$, is an irreducible finite-dimensional unitary representation of P , then

$$\mathrm{Ind}_P^G(\sigma^n \times \lambda^s)' \simeq \begin{cases} \pi(n, s) & \text{unless } n = 1/2, 3/2, \dots \text{ and } s = 0, \\ \pi^+(n, 1/2) \oplus \pi^-(n, 1/2) & \text{otherwise.} \end{cases}$$

(ii) If T^n is a generic representation of P for $n = 0, \pm 1/2, \pm 1, \dots$, then $\mathrm{Ind}_P^G T^n \simeq T_c \oplus T_d$, where T_c is a continuous direct sum with respect to Plancherel measure on $\hat{G}$ of representations $\pi(k, s)$ from the principal series of G with $k \geq |n|$, $k - n$ an integer, and if $n \geq 1$, T_d is a discrete direct sum of representations $\pi^+(k, s)$ from the discrete series of G with 1 or $3/2 \leq s \leq n \leq k$, $k - n$ an integer, while if $n \leq -1$, T_d is a discrete direct sum of representations $\pi^-(k, s)$ from the discrete series of G with 1 or $3/2 \leq s \leq -n \leq k$, $k - n$ an integer. The multiplicity of each irreducible representation appearing in $T_c \oplus T_d$ is one.

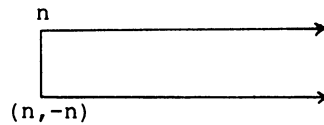
The results in part (i) of this theorem are classic. The results of part (ii) follow directly from Theorem 1 and the Mackey-Anh reciprocity theorem. Note that $\mathrm{Ind}_P^G T^n$ will contain only principal series representations in its decomposition for $n = 0, 1/2$ or $-1/2$ (i.e., $T_d = 0$). For $n > 1/2$, T_d contains discrete series representations of G from the positive side only, while for $n < -1/2$, T_d contains discrete series representations from the negative side only.

If we use the parametrization of $\hat{G}_p \cup \hat{G}_d$ as a subset of R^3 given in §2, then we can use the base of this fibre space to describe the decompositions in part (ii) of this theorem. For $n = 1, 3/2, \dots$, the representations appearing in the decomposition of $\mathrm{Ind}_P^G T^n$ will have a base space consisting of all lattice points (k, s) with $k - n \in Z$, $s \neq 1/2$, that also lie within a closed region of the form



while, for $n = -1, -3/2, \dots$, the representations appearing in this decomposition will have a base space consisting of all lattice points (k, s) with $k - n \in Z$, $s \neq -1/2$,

that also lie within a closed region of the form:



4. Decomposing tensor products. In this section we determine the decomposition of $\pi(n, s) \oplus \pi$ into irreducibles where $\pi(n, s)$ is a principal series representation of G and $\pi \in \hat{G}$ is arbitrary. We begin by using Mackey's tensor product theorem to write

$$\pi(n, s) \oplus \pi = \text{Ind}_P^G(\sigma^n \times \lambda^s)' \oplus \text{Ind}_G^G \pi \simeq \text{Ind}_P^G\{(\sigma^n \times \lambda^s)' \oplus (\pi)_P\}.$$

Thus, the problem of decomposing $\pi(n, s) \oplus \pi$ reduces to finding the decomposition $(\pi)_P$, finding the decomposition of $L = (\sigma^n \times \lambda^s)' \otimes (\pi)_P$ as a tensor product on P , and then finding the decomposition of $\text{Ind}_P^G L$ into irreducibles.

From §3, we know that, for each $\pi \in \hat{G}$, the restriction of π to the subgroup P , $(\pi)_P$, decomposes into a finite discrete sum of generic representations of P , each occurring with multiplicity one. Hence the problem of decomposing $L = (\sigma^n \times \lambda^s)' \otimes (\pi)_P$ can be further reduced to that of knowing how to decompose the tensor product of a finite-dimensional representation in P with an infinite-dimensional representation in $\hat{P}$. In Theorem 3, we use Mackey's tensor product theorem to show that the latter tensor product also decomposes into a finite sum of generic representations of P , each occurring with multiplicity 1. Thus, the representation L will decompose into a discrete direct sum of generic representations of P , each occurring with finite multiplicity. We describe the representations L for the various cases of $\pi \in \hat{G}$ in Theorem 4. Since $\pi(n, s) \otimes \pi \simeq \text{Ind}_P^G L$, the decomposition of $\pi(n, s) \oplus \pi$ into irreducibles can then be determined from a finite number of applications of Theorems 2 and 4. These results are presented in Theorem 5. Since the results are difficult to describe in closed form, we present tables which are useful to generate the multiplicities described in Theorem 5 for a given tensor product.

THEOREM 3. *If $(\sigma^n \times \lambda^s)'$ is a finite-dimensional irreducible unitary representation of P and T^k is an infinite-dimensional irreducible unitary representation of P , then*

$$(\sigma^n \times \lambda^s)' \otimes T^k \simeq T^{k-n} \oplus T^{k-n+1} \oplus \dots \oplus T^{k+n}.$$

PROOF. We again use Mackey's tensor product theorem to write

$$\begin{aligned} T^k \otimes (\sigma^n \times \lambda^s)' &= \text{Ind}_{TN}^P(\tau^k \times \beta_1) \otimes \text{Ind}_P^P(\sigma^n \times \lambda^s)' \\ &\simeq \text{Ind}_{TN}^P\{(\tau^k \times \beta_1) \otimes ((\sigma^n)_T \times I)\} \\ &\simeq \text{Ind}_{TN}^P\{\tau^k \otimes (\tau^{-n} \oplus \dots \oplus \tau^n) \times \beta_1\} \\ &\simeq T^{k-n} \oplus T^{k-n+1} \oplus \dots \oplus T^{k+n}. \end{aligned}$$

This theorem, in combination with the results of §3 on inducing from P and restricting to P , will enable us to determine the decomposition of $\pi(n, s) \otimes \pi$. We first describe this decomposition in terms of a discrete direct sum of representations of the form $\text{Ind}_P^G T$ where T is a generic representation in $\hat{P}$.

THEOREM 4. *Let $G = \text{Spin}(4, 1)$. The tensor product of the principal series representation $\pi(n, s)$ with the representation $\pi \in \bar{G}$ is unitarily equivalent to the representation $\text{Ind}_P^G L$ where L is a discrete direct sum of a finite number of generic representations of P occurring with finite multiplicities. The indices of the representations T^j appearing in L will have the form $-(m+n), -(m+n)+1, \dots, m+n$ when $\pi = \pi(m, \cdot)$ is a principal or complementary series representation, $-n, -n+1, \dots, n$ when π is an endpoint representation, and the form $q-n, q-n+1, \dots, m+n$ when $\pi = \pi^+(m, q)$. The range of the multiplicities of the representations appearing in L will always be symmetric, begin with one, increase by ones up to some value, remain constant for a segment, and then decrease from this constant value by ones back down to one. There are five cases to consider:*

1. If $\pi = \pi(m, \cdot)$ is a principal series representation with $m \geq n$, then L is

$$\sum_{j=0}^{2n} \oplus j T^{-(m+n)+(j-1)} \oplus (2n+1) \sum_{j=n-m}^{m-n} \oplus T^{m-m+j} \oplus \sum_{j=0}^{2n} \oplus j T^{(m+n)-(j-1)}.$$

2. If $\pi = \pi(m, \cdot)$ is a complementary series representation, then L is the same as the L obtained as in part 1 for the tensor product of the two principal series representations $\pi(n, s)$ and $\pi(m, 0)$.

3. If $\pi = \pi_{m,0}$ is an endpoint representation, then $\pi(n, s) \otimes \pi = \pi(n, s) \otimes \pi(0, 0) = \pi(0, 0) \otimes \pi(n, s)$ so that, as described in part 1, L is

$$T^{-n} \oplus T^{-n+1} \oplus \dots \oplus T^{n-1} \oplus T^n.$$

4. If $\pi = \pi^+(m, q)$ is a discrete or reducible principal series representation from the "positive side", then

(i) if $q+n \leq m-n$, then L is

$$\sum_{j=0}^{2n} \oplus j T^{(q-n)+(j-1)} \oplus (2n+1) \sum_{j=0}^{m-q-2n} \oplus T^{q+n+j} \oplus \sum_{j=0}^{2n} \oplus j T^{m+n-(j-1)};$$

(ii) while if $q+n > m-n$, then L is

$$\sum_{j=0}^{m-q} \oplus j T^{(q-n)+(j-1)} \oplus (m-q+1) \sum_{j=0}^{q+2n-m} \oplus T^{m-n+j} \oplus \sum_{j=0}^{m-q} \oplus j T^{m+n-(j-1)}.$$

5. If $\pi = \pi^-(m, q)$ is a discrete or reducible principal series representation from the "negative side", then L is equivalent to the representation obtained by changing each T^i to T^{-i} in the L of part 4 using $\pi^+(m, q)$.

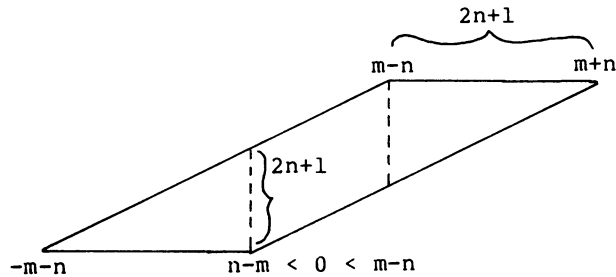
PROOF. Case 1. If $\pi = \pi(m, \cdot)$ is a principal series representation, then $(\pi)_P \simeq T^{-m} \oplus T^{-m+1} \oplus \dots \oplus T^m$ and hence

$$\begin{aligned} (\sigma^n \times \lambda^s)' \otimes (\pi)_P &\simeq \sum_{j=0}^{2m} \oplus \{(\sigma^n \times \lambda^s)' \otimes T^{-m+j}\} \\ &\simeq \sum_{j=0}^{2m} \sum_{i=0}^{2n} \oplus T^{-m-n+i+j} \end{aligned}$$

from Theorem 3. If $n = 0$, we have

$$\pi(0, s) \otimes \pi \simeq \text{Ind}_P^G(T^{-m} \oplus T^{-m+1} \oplus \cdots \oplus T^m).$$

If $n > 0$, the indices of the representations T^k appearing in this double summation form a lattice in the plane of the following type:



Since the multiplicity of a representation T^k appearing in the above double summation will be the number of lattice points on and within this parallelogram that also lie on the vertical line $x = k$, we see that the multiplicities will begin with 1 for T^{-m-n} , increase by 1's up to the value $2n + 1$ for T^{n-m} , will be $2n + 1$ until T^{m-n} , and will then decrease by 1's until we get to the multiplicity 1 for T^{m+n} .

Case 2. If $\pi = \pi(m, \cdot)$ is a complementary series representation, then the restriction of π to P is unitarily equivalent to the restriction to P of the principal series representation $\pi(m, 0)$. Thus, the tensor product $\pi(n, s) \otimes \pi$ will be equivalent to the tensor product of the two principal series representations $\pi(n, s)$ and $\pi(m, 0)$.

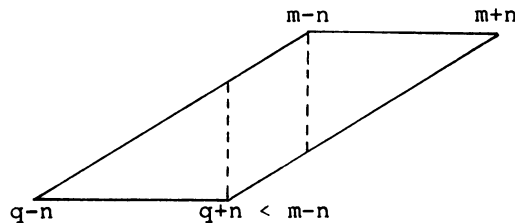
Case 3. If π is an endpoint representation, then the restriction to P of π is equivalent to the restriction to P of the class one principal series representation $\pi(0, 0)$, viz., T^0 .

Case 4. If $\pi = \pi^+(m, q)$, then $(\pi)_P \simeq \sum_{j=0}^{m-q} T^{q+j}$ and so

$$(\sigma^n \times \lambda^s)' \otimes (\pi)_P \simeq \sum_{j=0}^{m-q} \sum_{i=0}^{2n} T^{q-n+i+j}.$$

We now argue as in Case 1 by treating two subcases.

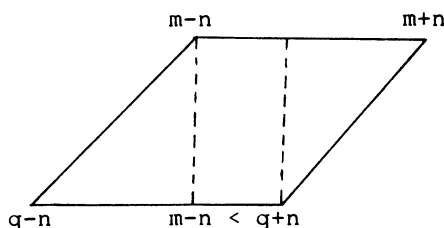
(i) If $q + n \leq m - n$, then for $n = 0$ we have $\pi(0, s) \otimes \pi \simeq T^q \oplus T^{q+1} \oplus \cdots \oplus T^m$ while for $n > 0$, the indices of the representations T^k appearing in the above double summation will form a lattice of the type:



Since the multiplicities, once again, will be the number of lattice points within the closed parallelogram that also lie on vertical lines, we see that the multiplicities

will begin with 1 for T^{q-n} , increase by 1's up to the value $2n+1$ for T^{q+n} , will be $2n+1$ until T^{m-n} , and will then decrease by 1's until we get the multiplicity 1 for T^{m+n} .

(ii) For $q+n > m-n$, our lattice will have the form



and so the multiplicities will begin with 1 for T^{q-n} , increase by 1's until they reach $m-q+1$ for T^{m-n} , remain $m-q+1$ until T^{q+n} , and then decrease by 1's until we get to 1 for T^{m+n} .

Case 5. If $\pi = \pi^-(m, q)$, then $(\pi)_P \simeq \sum_{j=0}^{m-q} \bigoplus T^{-m+j}$ and so

$$(\sigma^n \times \lambda^s)' \otimes (\pi)_P \simeq \sum_{j=0}^{m-q} \sum_{i=0}^{2n} \bigoplus T^{-m-n+i+j}.$$

We may now argue as in Case 4 or simply note that a representation T^k appearing in the above double summation will be the same as the multiplicity of the representation T^{-k} in the double summation of Case 4 with $\pi = \pi^+(m, q)$.

For example, we have: in the decomposition $\pi(2, \cdot) \otimes \pi(3, \cdot) = \text{Ind}_P^G L$, L will be the discrete direct sum of the representations $T^{-5}, T^{-4}, \dots, T^4, T^5$ occurring with the multiplicities 1, 2, 3, 4, 5, 5, 5, 4, 3, 2, 1, respectively; for the decomposition of $\pi(2, \cdot) \otimes \pi^+(9, 4)$, L will be the direct sum of $T^2, T^3, \dots, T^{10}$, and T^{11} with multiplicities 1, 2, 3, 4, 5, 5, 4, 3, 2, 1, respectively; for $\pi(2, \cdot) \otimes \pi^-(9, 4)$, L will be the direct sum of $T^{-11}, T^{-10}, \dots, T^{-3}$, and T^{-2} with multiplicities 1, 2, 3, 4, 5, 5, 4, 3, 2, 1, respectively; for $\pi(2, \cdot) \otimes \pi^+(11/2, 1/2)$, L will be the direct sum of $T^{-3/2}, T^{-1/2}, \dots, T^{13/2}$, and $T^{15/2}$ with multiplicities 1, 2, 3, 4, 5, 5, 4, 3, 2, 1, respectively; for $\pi(2, \cdot) \otimes \pi^+(4, 1)$, L will be the direct sum of $T^{-1}, T^0, \dots, T^5$, and T^6 with multiplicities 1, 2, 3, 4, 4, 3, 2, 1, respectively; for $\pi(3/2, \cdot) \otimes \pi(11/2, 7/2)$, L will be the sum of T^2, T^3, T^4, T^5, T^6 , and T^7 with multiplicities 1, 2, 3, 3, 2, 1, respectively; the decomposition of the tensor product of the principal series representation $\pi(n, \cdot)$ with $\pi^+(m, m)$ will have L equivalent to the sum of $T^{m-n}, T^{m-n+1}, \dots, T^{m+n}$ each occurring with multiplicity one (so that when $n = 0$, we get $L = T^m$); and $\pi(0, \cdot) \otimes \pi(m, \cdot)$ will have $L = T^{-m} \oplus \dots \oplus T^m$.

We are now ready to state the main theorem of this paper concerning the decomposition of $\pi(n, \cdot) \otimes \pi$. The multiplicities of the various representations occurring in this decomposition are difficult to describe in closed form. For a specific decomposition, they are somewhat easier to generate by using the tables we shall provide later. The results when π is a principal series representation of G were obtained in [13]. The results when π is a complementary series or endpoint representation can be obtained by replacing π with an appropriate principal series representation. The results when π is a discrete series representation from the positive side are described in 6 cases according to how the numbers $|q-n|, |m-n|$ and $q+n$ are

ordered in the reals. The results for $\pi = \pi^-(m, q)$ are phrased in terms of the results for $\pi^+(m, q)$.

THEOREM 5. *Let $\pi(n, \cdot)$ be a principal series representation of $G = \text{Spin}(4, 1)$ and $\pi \in \hat{G}$. Then $\pi(n, \cdot) \otimes \pi \simeq T_c \oplus T_d$, where T_c is a continuous direct sum with respect to Plancherel measure on $\hat{G}$ of representations from the principal series of G and T_d is a discrete direct sum of representations from the discrete series of G . The multiplicities of principal and discrete series representations appearing in this decomposition are all finite, are fixed from some point on (so that the range of the multiplicity function for this decomposition is finite), and depend only upon n and the restriction of π to the minimal parabolic subgroup P . There are ten cases to consider:*

A. *If $\pi = \pi(m, \cdot)$ is another principal series representation of G with $m \geq n$, then only representations $\pi(k, s)$ ($k \geq 0$ or $1/2$) from the principal series of G or representations $\pi^\pm(k, s)$ (with 1 or $3/2 \leq s \leq k$) from the discrete series of G with $k + m + n \equiv 0 \pmod{1}$ will occur in this decomposition; the multiplicity $m(k, s)$ of $\pi(k, s)$ in T_c will be*

$$\begin{aligned} (2k+1)(2n+1) & \quad \text{if } 0 \text{ or } 1/2 \leq k \leq m-n, \\ (2k+1)(2n+1) - j(j+1) & \quad \text{if } k = m-n+j, \quad j = 0, 1, 2, \dots, 2n, \\ (2m+1)(2n+1) & \quad \text{if } k \geq m+n; \end{aligned}$$

the multiplicity $m^+(k, s)$ of the discrete series representation $\pi^+(k, s)$ in T_d equals the multiplicity $m^-(k, s)$ of the discrete series representation $\pi^-(k, s)$ in T_d and will be

$$\begin{aligned} (k-s+1)(2n+1) & \quad \text{if } 1 \text{ or } 3/2 \leq s \leq k \leq m-n, \\ (k-s+1)(2n+1) - 1/2j(j+1) & \quad \text{if } k = m-n+j, \quad j = 0, 1, 2, \dots, 2n, \\ & \quad 1 \text{ or } 3/2 \leq s \leq m-n, \\ (k-s+1)(2n+1-h) & \quad \text{if } k = m-n+j, \quad j = 0, 1, 2, \dots, 2n, \\ -1/2(j-h)(j-h+1) & \quad s = m-n+h, \quad h = 0, 1, \dots, j, \\ m^\pm(k, s) = m^\pm(m+n, s) & \quad \text{if } 1 \text{ or } 3/2 \leq s \leq m+n \leq k, \\ 0 & \quad \text{if } m+n < s \leq k. \end{aligned}$$

B. *If $\pi = \pi(m, \cdot)$ is a representation in the complementary series of G , then $\pi(n, \cdot) \otimes \pi \simeq \pi(n, \cdot) \otimes \pi(m, 0)$, as in part A.*

C. *If $\pi = \pi_{m,0}$ is an endpoint representation of G , then $\pi(n, \cdot) \otimes \pi \simeq \pi(n, \cdot) \otimes \pi(0, 0)$, as in part A.*

D. *If $\pi = \pi^+(m, q)$ is a discrete series ($q > 1/2$) or a reducible principal series representation ($q = 1/2$) of G , then only representations $\pi(k, s)$ from the principal series of G (0 or $1/2 \leq k$) or representations $\pi^\pm(k, s)$ (1 or $3/2 \leq s \leq k$) from the discrete series of G with $k + m + n \equiv 0 \pmod{1}$ will occur in this decomposition.*

1. *If $q \geq n$, T_d will contain only discrete series representations from the positive side;*

(a) If $q \leq m - 2n$, then $0 \leq q - n \leq q + n \leq m - n \leq m + n$ and the multiplicity $m(k, s)$ of the principal series representation $\pi(k, s)$ in T_c will be

$$\begin{array}{ll} 0 & \text{if } 0 \text{ or } 1/2 \leq k < q - n, \\ 1/2(j+1)(j+2) & \text{if } k = q - n + j, \quad j = 0, 1, \dots, 2n, \\ (n+j+1)(2n+1) & \text{if } k = q + n + j, \quad j = 0, 1, \dots, m - q - 2n, \\ (k - q + 1)(2n + 1) - 1/2j(j+1) & \text{if } k = m - n + j, \quad j = 0, 1, 2, \dots, 2n, \\ (m - q + 1)(2n + 1) & \text{if } k \geq m + n; \end{array}$$

while the multiplicity $m^+(k, s)$ of the discrete series representation $\pi^+(k, s)$ in T_d will be

$$\begin{array}{ll} 0 & \text{if } 1 \text{ or } 3/2 \leq k < q - n \text{ or } m + n < s \leq k, \\ m^+(k, s) = m^+(k, q - n) & \text{if } 1 \text{ or } 3/2 \leq s \leq q - n \leq k, \\ m(k, s) - 1/2h(h+1) & \text{if } k \geq q - n \geq 3/2 \text{ or } 1, \\ & s = q - n + h, \quad h = 0, \dots, 2n, \\ m(k, s) - (n + h)(2n + 1) & \text{if } k \geq q + n + 1, \\ & s = q + n + h, \quad h = 0, 1, \dots, m - q - 2n, \\ (j - h + 1)(2n + 1) - 1/2j(j+1) & \text{if } k = m - n + j, \quad j = 0, 1, \dots, 2n, \\ + 1/2(h - 1)h & s = m - n + h, \quad h = 0, 1, \dots, j, \\ m^+(k, s) = m^+(m + n, s) & \text{if } 1 \text{ or } 3/2 \leq s \leq m + n \leq k; \end{array}$$

(b) if $q > m - 2n$, then $0 \leq q - n \leq m - n < q + n \leq m + n$ and the multiplicity $m(k, s)$ of the principal series representation $\pi(k, s)$ in T_c will be

$$\begin{array}{ll} 0 & \text{if } 0 \text{ or } 1/2 \leq k < q - n, \\ 1/2(j+1)(j+2) & \text{if } k = q - n + j, \quad j = 0, 1, \dots, m - q, \\ 1/2(m - q + 1)(m - q + 2j + 2) & \text{if } k = m - n + j, \\ & j = 0, 1, \dots, q + 2n - m, \\ 1/2(m - q + 1)(q - m + 4n + 2j + 2) & \text{if } k = q + n + j, \quad j = 0, 1, 2, \dots, m - q, \\ - 1/2j(j+1) & \\ (m - q + 1)(2n + 1) & \text{if } k \geq m + n; \end{array}$$

while the multiplicity $m^+(k, s)$ of the discrete series representation $\pi^+(k, s)$ in T_d will be

$$\begin{array}{ll} 0 & \text{if } 1 \text{ or } 3/2 \leq k < q - n \text{ or } \\ & m + n < s \leq k, \\ m^+(k, s) = m^+(k, q - n) & \text{if } 1 \text{ or } 3/2 \leq s \leq q - n \leq k, \\ m(k, s) - 1/2h(h+1) & \text{if } k \geq q - n \geq 3/2 \text{ or } 1, \\ & s = q - n + h, \quad h = 0, \dots, m - q, \\ m(k, s) - 1/2(m - q + 1)(m - q - 2h + 4), & \text{if } k \geq m - n + 1, \quad s = m - n + h, \\ & h = 0, 1, \dots, q + 2n - m, \\ (j - h + 1)(m - q + 1) - 1/2j(j+1) & \text{if } k = q + n + j, \quad j = 0, 1, \dots, m - q, \\ + 1/2(h - 1)h/h & s = q + n + h, \quad h = 0, 1, \dots, j, \\ m^+(k, s) = m^+(m + n, s) & \text{if } 1 \text{ or } 3/2 \leq s \leq m + n \leq k. \end{array}$$

2. If $q < n$, T_d will contain discrete series representations from both sides:

(a) if $q \leq m - 2n$, then $0 < n - q < n + q \leq m - n \leq m + n$ and the multiplicity $m(k, s)$ of the principal series representation $\pi(k, s)$ in T_c will be

$$\begin{array}{ll} (2k+1)(n-q+1) & \text{if } 0 \text{ or } 1/2 \leq k \leq n-q, \\ (2n-2q+1)(k+1) + 1/2j(j+1) & \text{if } k = n-q+j, \quad j = 0, 1, \dots, 2q, \\ (2n+1)(n+j+1) & \text{if } k = q+n+j, \\ & j = 0, 1, \dots, m-q-2n, \\ (2n+1)(m-q-n+j+1) - 1/2j(j+1) & \text{if } k = m-n+j, \quad j = 0, 1, 2, \dots, 2n, \\ (2n+1)(m-q+1) & \text{if } k \geq m+n; \end{array}$$

while the multiplicity $m^+(k, s)$ of the discrete series representation $\pi^+(k, s)$ in T_d will be

$$\begin{array}{ll} (k-s+1)(n-q+1+1/2(k+s)) & \text{if } 1 \text{ or } 3/2 \leq s \leq k \leq q+n, \\ (2n+1)(k-s+1) & \text{if } q+n < k \leq m-n, \quad q+n \leq s \leq k, \\ m^+(n+q, s) + (2n+1)(k-q-n) & \text{if } 1 \text{ or } 3/2 \leq s \leq q+n < k \leq m-n, \\ (k-s+1)(m+n+1-1/2(k+s)) & \text{if } m-n \leq s \leq k \leq m+n, \\ m^+(m-n, s) + m^+(k, m-n+1) & \text{if } 1 \text{ or } 3/2 \leq s \leq m-n < k \leq m+n, \\ m^+(m+n, s) & \text{if } 1 \text{ or } 3/2 \leq s \leq m+n \leq k, \\ 0 & \text{if } m+n < s \leq k; \end{array}$$

and the multiplicity $m^-(k, s)$ of the discrete series representation $\pi^-(k, s)$ in T_d will be

$$\begin{array}{ll} (k-s+1)(n-q+1-1/2(k+s)) & \text{if } 1 \text{ or } 3/2 \leq s \leq k \leq n-q, \\ 1/2(n-q-s+1)(n-q-s+2) & \text{if } 1 \text{ or } 3/2 \leq s \leq n-q \leq k, \\ 0 & \text{if } n-q < s \leq k; \end{array}$$

(b) if $q > m - 2n$ and $q \geq 2n - m$, then $0 < n - q \leq m - n < q + n \leq m + n$ and the multiplicity $m(k, s)$ of the principal series representation $\pi(k, s)$ in T_c will be

$$\begin{array}{ll} (2k+1)(n-q+1) & \text{if } 0 \text{ or } 1/2 \leq k \leq n-q, \\ (2n-2q+1)(k+1) + 1/2j(j+1) & \text{if } k = n-q+j, \quad j = 0, 1, \dots, m+q-2n, \\ 1/2(m-q+1)(m-q+2j+2) & \text{if } k = m-n+j, \\ & j = 0, 1, \dots, q+2n-m, \\ 1/2(m-q+1)(q-m+4n+2j+2) & \text{if } k = q+n+j, \quad j = 0, 1, 2, \dots, m-q, \\ & -1/2j(j+1) \\ (2n+1)(m-q+1) & \text{if } k \geq m+n; \end{array}$$

while the multiplicity $m^+(k, s)$ of the discrete series representation $\pi^+(k, s)$ in T_d will be

$$\begin{array}{ll} (k-s+1)(n-q+1+1/2(k+s)) & \text{if } 1 \text{ or } 3/2 \leq s \leq k \leq m-n, \\ (m-q+1)(k-s+1) & \text{if } m-n < k \leq q+n, \quad m-n \leq s \leq k, \\ m^+(m-n, s) + (m-q+1)(k-m+n) & \text{if } 1 \text{ or } 3/2 \leq s \leq m-n < k \leq q+n, \\ (k-s+1)(m+n+1-1/2(k+s)) & \text{if } q+n \leq s \leq k \leq m+n, \\ m^+(q+n, s) + m^+(k, q+n+1) & \text{if } 1 \text{ or } 3/2 \leq s \leq q+n < k \leq m+n, \\ m^+(m+n, s) & \text{if } 1 \text{ or } 3/2 \leq s \leq m+n \leq k, \\ 0 & \text{if } m+n < s \leq k; \end{array}$$

and the multiplicity $m^-(k, s)$ of the discrete series representation $\pi^-(k, s)$ in T_d will be

$$\begin{array}{ll} (k-s+1)(n-q+1-1/2(k+s)) & \text{if } 1 \text{ or } 3/2 \leq s \leq k \leq n-q, \\ 1/2(n-q-s+1)(n-q-s+2) & \text{if } 1 \text{ or } 3/2 \leq s \leq n-q \leq k, \\ 0 & \text{if } n-q < s \leq k; \end{array}$$

(c) if $m \geq n$ and $q < 2n-m$, then $0 \leq m-n \leq n-q < q+n \leq m+n$ and the multiplicity $m(k, s)$ of the principal series representation $\pi(k, s)$ in T_c will be

$$\begin{array}{ll} (2k+1)(n-q+1) & \text{if } 0 \text{ or } 1/2 \leq k \leq m-n, \\ (2k+1)(n-q+1) - 1/2j(j+1) & \text{if } k = m-n+j, \\ & j = 0, 1, \dots, 2n-m-q, \\ 1/2(m-q+1)(4n-m-3q+2j+2) & \text{if } k = n-q+j, j = 0, 1, \dots, 2q, \\ 1/2(m-q+1)(q-m+4n+2j+2) & \text{if } k = q+n+j, j = 0, 1, 2, \dots, m-q, \\ & - 1/2j(j+1) \\ (2n+1)(m-q+1) & \text{if } k \geq m+n; \end{array}$$

while the multiplicity $m^+(k, s)$ of the discrete series representation $\pi^+(k, s)$ in T_d will be

$$\begin{array}{ll} (k-s+1)(n-q+1+1/2(k+s)) & \text{if } 1 \text{ or } 3/2 \leq s \leq k \leq m-n, \\ (m-q+1)(k-s+1) & \text{if } m-n \leq k \leq q+n, m-n \leq s \leq k, \\ m^+(m-n, s) + (m-q+1)(k-m+n) & \text{if } 1 \text{ or } 3/2 \leq s \leq m-n \leq k \leq q+n, \\ (k-s+1)(m+n+1-1/2(k+s)) & \text{if } q+n \leq s \leq k \leq m+n, \\ m^+(q+n, s) + m^+(k, q+n+1) & \text{if } 1 \text{ or } 3/2 \leq s \leq q+n < k \leq m+n, \\ m^+(m+n, s) & \text{if } 1 \text{ or } 3/2 \leq s \leq m+n \leq k, \\ 0 & \text{if } m+n < s \leq k; \end{array}$$

and the multiplicity $m^-(k, s)$ of the discrete series representation $\pi^-(k, s)$ in T_d will be

$$\begin{array}{ll} (k-s+1)(n-q+1-1/2(k+s)) & \text{if } 1 \text{ or } 3/2 \leq s \leq k < n-q, \\ 1/2(n-q-s+1)(n-q-s+2) & \text{if } 1 \text{ or } 3/2 \leq s \leq n-q \leq k, \\ 0 & \text{if } n-q < s \leq k; \end{array}$$

(d) if $m < n$ and $q < 2n-m$, then $0 < n-m \leq n-q < q+n \leq m+n$ and the multiplicity $m(k, s)$ of the principal series representation $\pi(k, s)$ in T_c will be

$$\begin{array}{ll} (2k+1)(m-q+1) & \text{if } 0 \text{ or } 1/2 \leq k \leq n-m, \\ (2k+1)(m-q+1) - 1/2j(j+1) & \text{if } k = n-m+j, j = 0, 1, \dots, m-q, \\ 1/2(m-q+1)(4n-m-3q+2j+2) & \text{if } k = n-q+j, j = 0, 1, \dots, 2q, \\ 1/2(m-q+1)(q-m+4n+2j+2) & \text{if } k = q+n+j, j = 0, 1, 2, \dots, m-q, \\ & - 1/2j(j+1) \\ (2n+1)(m-q+1) & \text{if } k \geq m+n; \end{array}$$

while the multiplicity $m^+(k, s)$ of the discrete series representation $\pi^+(k, s)$ in T_d will be

$$\begin{array}{ll} (k-s+1)(m-q+1) & \text{if } 1 \text{ or } 3/2 \leq s \leq k \leq n+q, \\ (k-s+1)(m+n+1-1/2(k+s)) & \text{if } n+q \leq s \leq k \leq m+n, \\ m^+(q+n, s) + m^+(k, q+n+1) & \text{if } 1 \text{ or } 3/2 \leq s \leq q+n < k \leq m+n, \\ m^+(m+n, s) & \text{if } 1 \text{ or } 3/2 \leq s \leq m+n \leq k, \\ 0 & \text{if } m+n < s \leq k; \end{array}$$

and the multiplicity $m^-(k, s)$ of the discrete series representation $\pi^-(k, s)$ in T_d will be

$$\begin{array}{ll} (k-s+1)(m-q+1) & \text{if } 1 \text{ or } 3/2 \leq s \leq k \leq n-m, \\ (k-s+1)(n-q+1-1/2(k+s)) & \text{if } n-m \leq s \leq k \leq n-q, \\ m^-(n-m, s) + m^-(k, n-m+1) & \text{if } 1 \text{ or } 3/2 \leq s \leq n-m < k \leq n-q, \\ m^-(n-q, s) & \text{if } 1 \text{ or } 3/2 \leq s \leq n-q \leq k, \\ 0 & \text{if } n-q < s \leq k. \end{array}$$

E. If $\pi = \pi^-(m, q)$ is a discrete series ($q > 1/2$) or a reducible principal series ($q = 1/2$) representation of G , then the decomposition of $\pi(n, \cdot) \otimes \pi$ is obtained from part D by using $\pi^+(m, q)$ and interchanging the roles of $\pi^+(k, s)$ and $\pi^-(k, s)$.

Note. In cases A, B and C one need not have any discrete series representations occurring in the decomposition of the above tensor product (i.e., $T_d = 0$). In case A, for example, this will occur for $m = n = 0$ and $n = 0$, $m = 1/2$.

We will not provide a proof of this theorem in this paper. Instead, we will describe the process used to obtain the various multiplicities appearing in Theorem 5 since this process also provides a rather simple way of generating the multiplicities for a specific tensor product. The formulas appearing in Theorem 5 were obtained by treating each case separately and using tables similar to those discussed below. We will describe this process only for the cases $\pi = \pi(m, \cdot)$ and $\pi = \pi^+(m, q)$, since all other cases can be obtained from them.

We begin by drawing three horizontal lines. In the space between the two bottom lines we place the numbers 0 or $1/2, \dots, m+n$ (for $\pi = \pi(m, \cdot)$ or $\pi = \pi^+(m, q)$ with $n > q$) or $q-n, q-n+1, \dots, m+n$ (for $\pi = \pi^+(m, q)$ with $n \leq q$). These numbers will represent the k -index for the representations $\pi(k, s)$ or $\pi^\pm(k, s)$ appearing in our decomposition. Since all multiplicities for $k > m+n$ ($k+m+n \equiv 0 \pmod{1}$) will be the same as the multiplicity for $k = m+n$, we place the three dots after the $m+n$.

Motivated by the parametrization of $\hat{G}_p \cup \hat{G}_d$ as a fibre space described in §2, we will place the multiplicities of discrete series representations $\pi^\pm(k, s)$ ($s \geq 1$ or $3/2$) appearing in our decomposition above (for $\pi^+(k, s)$) and below (for $\pi^-(k, s)$) these lines using their natural parameters (k, s) . Since the multiplicities of principal series representations $\pi(k, s)$ on a given fiber (i.e., for a fixed k) are all the same, we will place this common multiplicity between the top two lines above the appropriate value of k , i.e., at $(k, 0)$.

All the multiplicities appearing in these tables can now be generated recursively from the multiplicities given in Theorem 4. We begin by taking these multiplicities and placing them in order along the two main diagonals in our table beginning

at $(m+n, m+n)$ in the top (positive side) of our table. If $\pi = \pi^+(m, q)$ with $n \leq q$, these numbers will occupy diagonal entries on the positive side of our table only except when $q - n = 0$ or $1/2$ —in this case, we place the remaining 1 between the horizontal lines (i.e., at either $(0, 0)$ or $(1/2, 0)$). For a principal series representation $\pi(m, \cdot)$ or a representation $\pi^+(m, q)$ with $n > q$, these numbers will occupy diagonal entries on both sides; for $m+n \equiv 0 \pmod{1}$, we place the multiplicity of T^0 between our horizontal lines at $(0, 0)$ while for $m+n \equiv 1/2 \pmod{1}$, we place the sum of the multiplicities for $T^{1/2}$ and $T^{-1/2}$ between the horizontal lines at $(1/2, 0)$. The remaining multiplicities in our table can now be determined from these diagonal entries one column at a time proceeding from left to right. To obtain the multiplicities in a given column in the positive or negative side one adds the diagonal entry of that side to all the entries in the preceding column on that side. The principal series multiplicity in a given column is obtained by adding both diagonal entries to the preceding principal series multiplicity.

For example, the table for $\pi(2, \cdot) \otimes \pi(3, \cdot)$ would be:

	5						1	...	
	4					2	3	...	
+	3				3	5	6	...	
	2			4	7	9	10	...	
s	1		5	9	12	14	15	...	
			5	15	23	29	33	35	...
		k	0	1	2	3	4	5	...
s	1			5	9	12	14	15	...
	2				4	7	9	10	...
-	3					3	5	6	...
	4						2	3	...
	5							1	...

Since tables for the tensor product of two principal series representations will be symmetric, we could also present tables for this case as in [13].

The table for $\pi(3, 2, \cdot) \otimes \pi^+(11/2, 7/2)$ would be:

	7						1	...
	6					2	3	...
	5				3	5	6	...
+	4			3	6	8	9	...
	3		2	5	8	10	11	...
s	2	1	3	6	9	11	12	...
		1	3	6	9	11	12	...
k		2	3	4	5	6	7	...

The table for $\pi(2, \cdot) \otimes \pi^+(11/2, 1/2)$ would be:

	15/2							1	...
	13/2						2	3	...
+	11/2					3	5	6	...
	9/2				4	7	9	10	...
	7/2			5	9	12	14	15	...
	5/2		5	10	14	17	19	20	...
s	3/2	4	9	14	18	21	23	24	...
		5	10	15	20	24	27	29	30
k		1/2	3/2	5/2	7/2	9/2	11/2	13/2	15/2
-	3/2	1	1	1	1	1	1	1	...

The table for $\pi(2, \cdot) \otimes \pi^-(11/2, 1/2)$ can be obtained from the table for $\pi(2, \cdot) \otimes \pi^+(11/2, 1/2)$ by interchanging the roles of + and - in the above table (for example, by interchanging the plus sign on the left side of the above table with the minus sign).

BIBLIOGRAPHY

1. N. Anh, *Restriction of the principal series of $SL(n, C)$ to some reductive sub-groups*, Pacific J. Math. **38** (1971), 295-313.
2. R. Boyer and R. Martin, *The group C^* -algebra of the DeSitter group*, Proc. Amer. Math. Soc. **65** (1977), 177-184.
3. A. Breda, *Uniformly bounded representations of the Lorentz groups*, Ph.D. Dissertation, Washington University, 1982.

4. J. Dixmier, *Représentations intégrables du groupe de DeSitter*, Bull. Soc. Math. France **89** (1961), 9–41.
5. V. K. Dobrev, G. Mack, V. B. Petkova, S. G. Petrova and I. T. Todorov, *Harmonic analysis on the n -dimensional Lorentz group and its application to conformal quantum field theory*, Lecture Notes in Phys., Vol. 63, Springer-Verlag, Berlin, Heidelberg and New York, 1977.
6. R. Fabec, *Localizable representations of the DeSitter group*, J. Analyse Math. **35** (1979), 151–208.
7. A. Kleppner and R. L. Lipsman, *The Plancherel formula for group extensions*, Ann. Sci. École Norm. Sup. (4) **5** (1972), 495–516.
8. A. W. Knap and E. M. Stein, *Intertwining operators for semi-simple groups*, Ann. of Math. (2) **93** (1971), 489–578.
9. R. A. Kunze and E. M. Stein, *Uniformly bounded representations. III: Intertwining operators for the principal series on semi-simple groups*, Amer. J. Math. **89** (1967), 385–442.
10. G. Mackey, *Induced representations of locally compact groups. II. The Frobenius reciprocity theorem*, Ann. of Math. (2) **58** (1953), 193–221.
11. R. Martin, *On the decomposition of tensor products of principal series representations for real-rank one semisimple groups*, Trans. Amer. Math. Soc. **201** (1975), 177–211.
12. ———, *Tensor products for $SL(2, k)$* , Trans. Amer. Math. Soc. **239** (1978), 197–211.
13. ———, *Tensor products of principal series for the DeSitter group*, Trans. Amer. Math. Soc. **265** (1981), 121–135.
14. L. Pukánszky, *On the Kronecker products of irreducible representations of the 2×2 real unimodular group. I*, Trans. Amer. Math. Soc. **100** (1961), 116–152.
15. J. Repka, *Tensor products of unitary representations of $SL_2(R)$* , Amer. J. Math. **100** (1978), 747–774.
16. H. Rossi and M. Vergne, *Representations of certain solvable Lie groups on Hilbert spaces of holomorphic functions and the application to the holomorphic discrete series of a semi-simple Lie group*, J. Funct. Anal. **13** (1973), 324–389.
17. R. Takahashi, *Sur les représentations unitaires des groupes de Lorentz généralisés*, Bull. Soc. Math. France **91** (1963), 289–433.
18. E. Thieleker, *The unitary representations of the generalized Lorentz groups*, Trans. Amer. Math. Soc. **199** (1974), 327–367.
19. N. J. Vilenkin, *Special functions and the theory of group representations*, Transl. Math. Monographs, Vol. 22, Amer. Math. Soc., Providence, R.I., 1968.
20. G. Warner, *Harmonic analysis on semi-simple Lie groups*, Vols. 1, 2, Springer-Verlag, Berlin, Heidelberg and New York, 1972.
21. F. Williams, *Tensor products of principal series representations*, Lecture Notes in Math., vol. 358, Springer-Verlag, Berlin and New York, 1973.

DEPARTMENT OF MATHEMATICS, MIDDLEBURY COLLEGE, MIDDLEBURY, VERMONT
05753